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NON-LINEAR TRANSIENT ANALYSIS OF MODERATELY THICK LAMINATED COMPOSITE PLATES

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The non-linear transient analysis of the shear deformable laminated composite plates, subjected to step, ramp and sinusoidal loading is presented. The clamped, simply supported, free and their combinations (non-Levy type) of boundary conditions are considered. The formulation is based on the Mindlin first order shear deformation theory and von-Karman non-linearity. The methodology of the solution is based on the Chebyshev series technique. Houbolt time marching scheme and quadratic extrapolation technique are used for the temporal discretization and linearization respectively. The effects of magnitude and duration of loading, rotary inertia, in-plane, inertia, b/a, a/h, lamination scheme and boundary conditions on the central displacement response are studied. Typical results are presented in dimensionless graphical forms for different parameters and loading conditions.

1. INTRODUCTION

The applications of laminated composite plates have increased manifold in the aerospace, aircraft and automobile industries. The dynamic behaviour of these structural elements at large amplitudes of motion in response to the conditions they are subjected to, have received considerable attention in the recent past. A comprehensive survey of non-linear vibration analysis of plates using approximate analytical and finite-element methods (FEMS) is presented by Sathyamoorthy [1]. There is an excellent review on the vibrations of plates due to Yamada and Irie [2]. Reddy [3] carried out the geometrically non-linear transient analysis of laminated composite plates undergoing moderately large deformations using FEM and the effects of plate thickness, lamination scheme, boundary conditions and loading on the deflections and stresses were investigated. Incorporating the generalized non-linearity, Bhimaraddi [4] studied the non-linear free vibration of simply supported shear deformable composite laminated plates using a regular perturbation technique and concluded that the formulation based on von-Karman-type non-linearity is quite satisfactory. The large amplitude-free vibration analysis of composite simply supported plate, using a multimode time-domain modal formulation based on the FEM is carried out by Shi *et al.* [5]. Employing von-Karman-type non-linearity, Carerra and Krause [6] studied dynamics of multi-layered thick plates using FEA with Newmark time-integration method. Effects of different geometries, loading conditions and layouts on the response are analyzed and it is found that in the case of unsymmetrically laminated plates, the linear

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solution overestimates or underestimates the non-linear results, depending on both the magnitude and type of the applied loading. Cheng *et al.* [7] studied the influence of transverse shear, rotary inertia and coupling of modes on the non-linear frequency of moderately thick rectangular plates and sandwich plates using harmonic balance and mixed Fourier series method. It is shown that the effect of rotatory inertia is of less importance and the non-linear fundamental frequency increases with the central amplitude (hardening type of non-linearity). Khdeir and Reddy [8] employed generalized Levy-type solutions with the state-space concept for the free vibration behaviour of cross-ply and antisymmetric angle-ply laminated plates and concluded that all shear deformation theories (first order, second order, third order) give matching frequencies for thick laminates, different from those of classical theory. Tuomala and Mikkola [9] used FEM to obtain the transient response of plates subjected to impulsive loading, taking into account the influences of geometry changes and material non-linearities. Non-linear oscillatory behaviour of unsymmetrical laminated plates is investigated by Singh and Rao [10] using FEM and employing four-node shear flexible composite plate element with six-degrees-of-freedom per node. The spectral analysis was used by Kadiri *et al.* [11] to obtain the second non-linear mode of a fully clamped rectangular plate and they concluded that the non-linear mode shows a higher bending stress close to the clamps at large deflections, compared with that predicted by linear theory.

The computational power of numerical methods such as the FEM and the finite-difference method is well established for complicated loading and geometry. These numerical methods are employed as a necessity because it is extremely difficult to solve analytically, the non-linear equations of motion of moderately thick laminated plate subjected to transient loading. For obtaining the global response of these structures with regular boundaries, analytical methods may turn out to be superior to numerical methods. Hence, there is a need to develop and explore the possibility of using analytical techniques for non-linear problems of laminated composite plates with non-classical edge restraints. The analytical technique used so far is applicable only to Levy-type boundary conditions. In the present paper, an attempt is made to employ the fast-converging double Chebyshev series for the non-linear analysis of moderately thick shear deformable laminated composite rectangular plates with non-classical boundary conditions, subjected to transient loadings. Kapania and Lovejoy [12] employed the Chebyshev series for the estimation of linear frequency and mode shapes of laminated quadrilateral plates. Nath and Sandeep [13] used Chebyshev polynomials for the non-linear static and dynamic analysis of thin rectnagular isotropic plates, employing Kirchhoff's hypothesis. Recently, the present authors [14] employed the double Chebyshev series for the non-linear static analysis of moderately thick laminated composite plates with different boundary conditions employing von-Karman-type kinematics.

In the present paper, the formulation is based on the von-Karman-type kinematics and first order shear deformation theory. The governing non-linear-coupled partial differential equations of motion are linearized by quadratic extrapolation technique. The double Chebyshev series and Houbolt time-marching scheme are used for spatial and temporal discretizations respectively. Convergence study was carried out and the results were validated with some of the published results. Laminated plates with different boundary conditions, consisting of clamped, simply supported, free and their combinations are considered under step functions, ramp and sinusoidal loading respectively. The effects of in-plane inertia, rotary inertia, aspect ratio, span-to-thickness ratio and lamination scheme on the large amplitude response are studied. Some typical results are reported here.

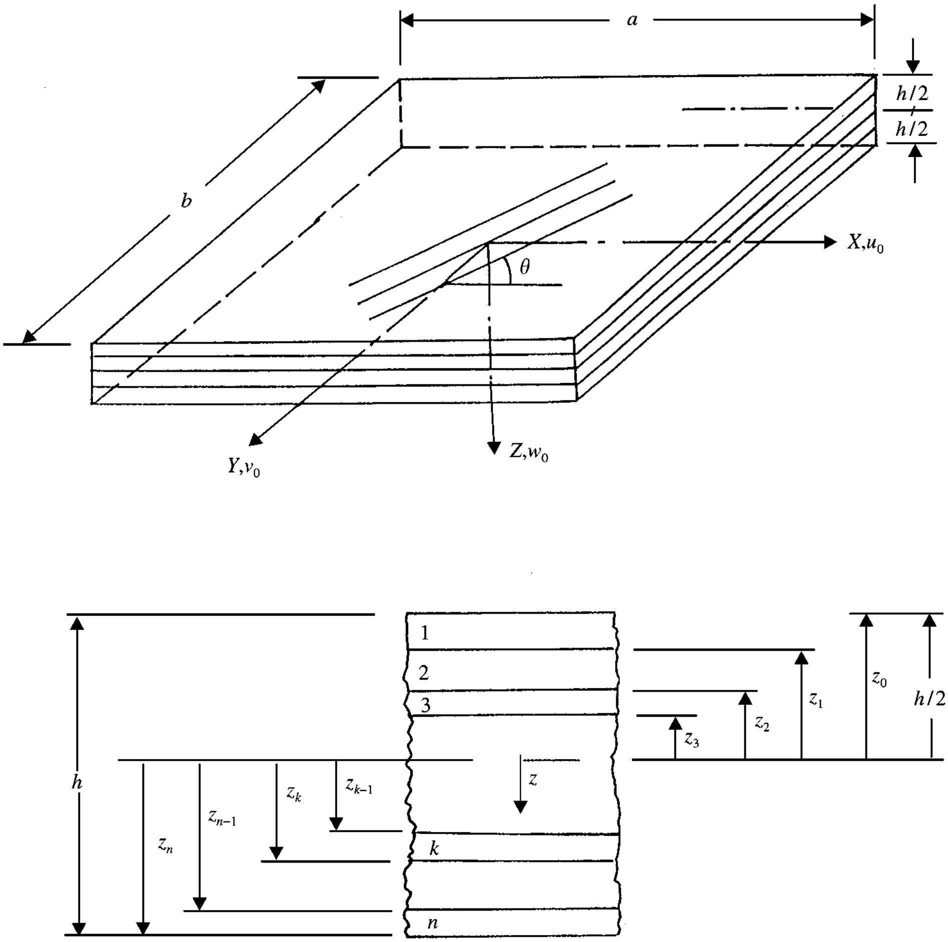

Figure 1. Geometry of laminated plate.

2. MATHEMATICAL FORMULATION

2.1. GOVERNING EQUATIONS

Figure 1 shows the geometry of the plate. Perfect bonding between the orthotropic layers is assumed. The displacement field at a point in the laminate is expressed as

$$U(x, y, z) = u_0(x, y) + z\psi_x(x, y),$$

$$V(x, y, z) = v_0(x, y) + z\psi_y(x, y),$$

$$W(x, y, z) = w_0(x, y). \tag{1}$$

Based on the von-Karman-type non-linearity, the strain–displacement relations become

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix} = \begin{Bmatrix} u_{0,x} + \frac{1}{2}(w_{0,x})^2 \\ v_{0,y} + \frac{1}{2}(w_{0,y})^2 \\ u_{0,y} + v_{0,x} + w_{0,x}w_{0,y} \\ w_{0,x} + \psi_x \\ w_{0,y} + \psi_y \end{Bmatrix} + z \begin{Bmatrix} \psi_{x,x} \\ \psi_{y,y} \\ \psi_{x,y} + \psi_{y,x} \\ 0 \\ 0 \end{Bmatrix}. \tag{2}$$

The stress and moment resultants of a laminated composite rectangular plate having n layers of orthotropic lamina can be expressed as

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} \\ B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{Bmatrix} u_{0,x} + \frac{1}{2}(w_{0,x})^2 \\ v_{0,y} + \frac{1}{2}(w_{0,y})^2 \\ u_{0,y} + v_{0,x} + w_{0,x}w_{0,y} \\ \psi_{x,x} \\ \psi_{y,y} \\ \psi_{x,y} + \psi_{y,x} \end{Bmatrix}, \tag{3}$$

$$\begin{Bmatrix} Q_y \\ Q_x \end{Bmatrix} = \begin{bmatrix} A_{44} & A_{45} \\ A_{45} & A_{55} \end{bmatrix} \begin{Bmatrix} \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix}, \tag{4}$$

where the laminate stiffness coefficients (A_{ij}, B_{ij}, D_{ij}) are defined in terms of the reduced stiffness coefficients $(Q_{ij})_k$ for the layers $k = 1, 2, \ldots, n$ [15] as

$$(A_{ij}, B_{ij}, D_{ij}) = \sum_{k=1}^{n} \int_{z_{k-1}}^{z_k} (1, z, z^2)\,(\bar{Q}_{ij})_k \,\mathrm{d}z \quad (i, j = 1, 2, 6), \tag{5}$$

$$A_{ij} = \sum_{k=1}^{n} k_i k_j \int_{z_{k-1}}^{z_k} (\bar{Q}_{ij})_k \,\mathrm{d}z, \quad (i, j = 4, 5), \tag{6}$$

where $k_4^2 = \frac{5}{6}$, $k_5^2 = \frac{5}{6}$ are shear correction factors [16].

Neglecting the body moments and surface shearing forces, the equations of motion [3] reduce to

$$N_{x,x} + N_{xy,y} = P\,u_{,tt} + R\,\psi_{x,tt}, \tag{7}$$

$$N_{xy,x} + N_{y,y} = P\,v_{,tt} + R\,\psi_{y,tt}, \tag{8}$$

$$Q_{x,x} + Q_{y,y} + (N_x\,w_{,x})_{,x} + (N_y\,w_{,y})_{,y} + (N_{xy}\,w_{,x})_{,y} + (N_{xy}\,w_{,y})_{,x} + q(x, y, t) = P\,w_{,tt}, \tag{9}$$

$$M_{x,x} + M_{xy,y} - Q_x = R\,u_{,tt} + I\,\psi_{x,tt}, \tag{10}$$

$$M_{xy,x} + M_{y,y} - Q_y = R\,v_{,tt} + I\,\psi_{y,tt}, \tag{11}$$

where P, R and I are the normal, coupled normal rotary and rotary inertia coefficients respectively. They are defined as

$$(P, R, I) = \int_{-h/2}^{h/2} (1, z, z^2)\,\rho\,\mathrm{d}z, \tag{12}$$

Using equations (2)–(6), equations (7)–(11) are transformed to the following non-dimensional forms:

$$\begin{aligned} &u_{,xx} + L_1\,u_{,yy} + L_2\,u_{,xy} + L_3\,v_{,xx} + L_4\,v_{,yy} + L_5\,v_{,xy} + L_6\,\psi_{x,xx} + L_7\psi_{x,yy} + L_8\psi_{x,xy} + L_9\psi_{y,xx} \\ &\quad + L_{10}\psi_{y,yy} + L_{11}\,\psi_{y,xy} + (L_{12}\,w_{,xx} + L_{13}\,w_{,yy} + L_{14}\,w_{,xy})\,w_{,x} + (L_{15}\,w_{,xx} + L_{16}\,w_{,yy} \\ &\quad + L_{17}\,w_{,xy})\,w_{,y} = F_1\,u_{,\tau\tau} + F_2\,\psi_{x,\tau\tau}, \end{aligned} \tag{13}$$

$$L_{18}u_{,xx}+L_{19}u_{,yy}+L_{20}u_{,xy}+L_{21}v_{,xx}+L_{22}v_{,yy}+L_{23}v_{,xy}+L_{24}\psi_{x,xx}+L_{25}\psi_{x,yy}+L_{26}\psi_{x,xy}$$
$$+L_{27}\psi_{y,xx}+L_{28}\psi_{y,yy}+L_{29}\psi_{y,xy}+(L_{30}w_{,xx}+L_{31}w_{,yy}+L_{32}w_{,xy})w_{,x}+(L_{33}w_{,xx}$$
$$+L_{34}w_{,yy}+L_{35}w_{,xy})w_{,y}=v_{,\tau\tau}+F_3\psi_{y,\tau\tau}, \tag{14}$$

$$L_{36}w_{,xx}+L_{37}w_{,yy}+L_{38}w_{,xy}+L_{39}\psi_{x,x}+L_{40}\psi_{x,y}+L_{41}\psi_{y,x}+L_{42}\psi_{y,y}+(L_{43}u_{,x}+L_{44}u_{,y}$$
$$+L_{45}v_{,x}+L_{46}v_{,y}+L_{47}\psi_{x,x}+L_{48}\psi_{x,y}+L_{49}\psi_{y,x}+L_{50}\psi_{y,y})w_{,xx}+(L_{51}u_{,x}+L_{52}u_{,y}$$
$$+L_{53}v_{,x}+L_{54}v_{,y}+L_{55}\psi_{x,x}+L_{56}\psi_{x,y}+L_{57}\psi_{y,x}+L_{58}\psi_{y,y})w_{,yy}+(L_{59}u_{,x}+L_{60}u_{,y}$$
$$+L_{61}v_{,x}+L_{62}v_{,y}+L_{63}\psi_{x,x}+L_{64}\psi_{x,y}+L_{65}\psi_{y,x}+L_{66}\psi_{y,y})w_{,xy}+(L_{67}w_{,xx}+L_{68}w_{,yy}$$
$$+L_{69}w_{,xy})(w_{,x})^2+(L_{70}w_{,xx}+L_{71}w_{,yy}+L_{72}w_{,xy})(w_{,y})^2+(L_{73}w_{,xx}+L_{74}w_{,yy}$$
$$+L_{75}w_{,xy})w_{,x}w_{,y}+q^*=w_{,\tau\tau}, \tag{15}$$

$$L_{76}u_{,xx}+L_{77}u_{,yy}+L_{78}u_{,xy}+L_{79}v_{,xx}+L_{80}v_{,yy}+L_{81}v_{,xy}+\psi_{x,xx}+L_{82}\psi_{x,yy}+L_{83}\psi_{x,xy}$$
$$+L_{84}\psi_{y,xx}+L_{85}\psi_{y,yy}+L_{86}\psi_{y,xy}+L_{87}w_{,x}+L_{88}w_{,y}+L_{89}\psi_x+L_{90}\psi_y+(L_{91}w_{,xx}$$
$$+L_{92}w_{,yy}+L_{93}w_{,xy})w_{,x}+(L_{94}w_{,xx}+L_{95}w_{,yy}+L_{96}w_{,xy})w_{,y}=F_4u_{,\tau\tau}+F_5\psi_{x,\tau\tau}, \tag{16}$$

$$L_{97}u_{,xx}+L_{98}u_{,yy}+L_{99}u_{,xy}+L_{100}v_{,xx}+L_{101}v_{,yy}+L_{102}v_{,xy}+L_{103}\psi_{x,xx}+L_{104}\psi_{x,yy}$$
$$+L_{105}\psi_{x,xy}+L_{106}\psi_{y,xx}+L_{107}\psi_{y,yy}+L_{108}\psi_{y,xy}+L_{109}w_{,x}+L_{110}w_{,y}+L_{111}\psi_x$$
$$+L_{112}\psi_y+(L_{113}w_{,xx}+L_{114}w_{,yy}+L_{115}w_{,xy})w_{,x}+(L_{116}w_{,xx}+L_{117}w_{,yy}+L_{118}w_{,xy})$$
$$\times w_{,y}=F_6v_{,\tau\tau}+F_7\psi_{y,\tau\tau}. \tag{17}$$

The non-dimensional parameters, $L_1, L_2, \cdots$ and $F_1, F_2, \cdots$ are given in Appendix A.

2.2. BOUNDARY CONDITIONS

(a) *Simply supported* (S):

$$u, v, w, \bar{M}_x, \psi_y = 0 \quad \text{at } x = -1 \text{ and } 1,$$
$$u, v, w, \psi_x, \bar{M}_y = 0 \quad \text{at } y = -1 \text{ and } 1;$$

(b) *Clamped* (C):

$$u, v, w, \psi_x, \psi_y = 0 \quad \text{at } x = -1 \text{ and } 1 \quad \text{and} \quad y = -1 \text{ and } 1;$$

(c) *Free* (F):

$$\bar{N}_x, \bar{N}_{xy}, \bar{M}_x, \bar{M}_{xy}, \bar{Q}_x = 0 \quad \text{at} \quad x = -1 \text{ and } 1,$$
$$\bar{N}_{xy}, \bar{N}_y, \bar{M}_y, \bar{M}_{xy}, \bar{Q}_y = 0 \quad \text{at} \quad y = -1 \text{ and } 1.$$

3. METHOD OF SOLUTION

A general function $\phi(x, y)$ can be approximated in space domain by finite-degree double Chebyshev polynomial [17] as

$$\phi(x, y) = \delta \sum_{i=0}^{M} \sum_{j=0}^{N} \phi_{ij} T_i(x) T_j(y), \tag{18}$$

where

$$\delta = \begin{cases} \frac{1}{4} & \text{if } i = 0 \quad \text{and} \quad j = 0, \\ \frac{1}{2} & \text{if } i = 0 \quad \text{and} \quad j \neq 0 \quad \text{or} \quad i \neq 0 \quad \text{and} \quad j = 0, \\ 1 & \text{otherwise.} \end{cases}$$

The spatial derivative of a general function $\phi(x, y)$ can be expressed as

$$\phi^{rs}(x, y) = \delta \sum_{i=0}^{M-r} \sum_{j=0}^{N-s} \phi_{ij}^{rs} T_i(x) T_j(y), \tag{19}$$

where r and s are the order of derivatives with respect to x and y respectively. The derivative function ϕ_{ij}^{rs} is evaluated using the following recurrence relations [17]:

$$\phi_{(i-1)j}^{rs} = \phi_{(i+1)j}^{rs} + 2i\phi_{ij}^{(r-1)s}, \qquad \phi_{i(j-1)}^{rs} = \phi_{i(j+1)}^{rs} + 2j\phi_{ij}^{r(s-1)}. \tag{20}$$

Non-linearity in the governing equations appears due to the product of the dependent variables. The non-linear terms are linearized at any step of marching variable using quadratic extrapolation technique. A typical non-linear function G at step J is expressed as

$$G_J = \left[\delta \sum_{i=0}^{M-r} \sum_{j=0}^{N} \phi_{ij}^{r} T_i(x) T_j(y) \right]_J * \left[\delta \sum_{i=0}^{M} \sum_{j=0}^{N-s} \phi_{ij}^{s} T_i(x) T_j(y) \right]_J, \tag{21}$$

where

$$(\phi_{ij})_J = A(\phi_{ij})_{J-1} + B(\phi_{ij})_{J-2} + C(\phi_{ij})_{J-3}.$$

During initial steps of marching variables, the coefficients A, B, C of the quadratic extrapolation scheme of linearization [13] take the following values:

$$1, 0, 0 \ (J = 1), \qquad 2, -1, 0 \ (J = 2), \qquad 3, -3, 1 \ (J \geqslant 3).$$

The product of two Chebyshev polynomials is expressed as

$$T_i(x) T_j(y) T_k(x) T_l(y) = [T_{i+k}(x) T_{j+l}(y) + T_{i+k}(x) T_{j-l}(y) + T_{i-k}(x) T_{j+l}(y) + T_{i-k}(x) T_{j-l}(y)]/4. \tag{22}$$

The displacement functions and loading are approximated by finite-degree Chebyshev polynomials as

$$(u, v, w, \psi_x, \psi_y, Q) = \delta \sum_{i=0}^{M} \sum_{j=0}^{N} (u_{ij}, v_{ij}, w_{ij}, \psi_{xij}, \psi_{yij}, Q_{ij}) T_i(x) T_j(y), \quad -1 \leqslant x \leqslant 1, \quad -1 \leqslant y \leqslant 1. \tag{23}$$

Implicit Houbolt time-marching method [18] is used to evaluate the acceleration terms $(u_{,\tau\tau})$, $(v_{,\tau\tau})$, $(w_{,\tau\tau})$, $(\psi_{x,\tau\tau})$ and $(\psi_{y,\tau\tau})$ in the governing equations (13)–(17). At time-step J, the general acceleration $(\phi_{,\tau\tau})_J$ is evaluated as

$$(\phi_{,\tau\tau})_J = (\beta_1 \phi_J + \beta_2 \phi_{J-1} + \beta_3 \phi_{J-2} + \beta_4 \phi_{J-3} + \beta_5)/(\Delta\tau^2). \tag{24}$$

Houbolt coefficients $\beta_i (i = 1, 5)$ can be evaluated during initial steps by using recurrence relations [18] and are given in Appendix B.

Using the above procedure of spatial discretization, linearization and temporal discretization, the non-linear differential equations (13)–(17) are discretized in space and time domains respectively. The set of generating linear algebraic equations becomes

$$\sum_{i=0}^{M-2} \sum_{j=0}^{N-2} \sum_{k=1}^{5} F_k (u_{ij}, v_{ij}, w_{ij}, \psi_{xij}, \psi_{yij}, Q_{ij}) \ T_i(x) T_j(y) = 0. \tag{25}$$

Similarly, the appropriate set of boundary conditions are also discretized.

TABLE 1

Convergence study of [0/90/0/90] *square plate* ($\beta = 10$, $Q = 50$)

		Space-wise convergence ($\Delta\tau = 0{\cdot}1$)			
		Centre			
Boundary conditions	M, N	w (max)	τ	M_x (max)	τ
CSCS	6	0·516899	11·2	2·52769	9·7
	7	0·493602	11·2	2·39125	9·8
	8	0·499053	11·4	2·29421	10·4
	9	0·497812	11·4	2·28463	10·3
	10	0·492611	11·1	2·36994	9·9
	11	0·492311	11·1	2·35379	9·9
CCCF	6	0·473765	12·6	2·12140	9·9
	7	0·463553	12·2	1·97009	9·9
	8	0·466490	12·0	1·98126	10·4
	9	0·467887	11·9	1·94334	10·5
	10	0·464364	12·1	1·98674	9·9
	11	0·464456	12·2	1·95503	10·0
		Time-wise convergence ($M = N = 9$)			
		Centre			
Boundary conditions	$\Delta\tau$	w (max)	τ	M_x (max)	τ
CSCS	0·2	0·474231	11·6	2·28616	10·4
	0·1	0·497812	11·4	2·28463	10·3
	0·05	0·497660	11·3	2·27933	10·6
CCCF	0·2	0·468474	12·2	1·895166	10·8
	0·1	0·467887	11·9	1·94334	10·5
	0·05	0·467853	11·85	1·922414	10·75

The total number of unknown coefficients in equation (23) are $5(M+1)\ (N+1)$. Collocating the zeros of Chebyshev polynomials, $5(M-1)(N-1)$ algebraic equations are generated from equation (25). Similarly, the SSSS, CSCS, CCCF, CCFF boundary conditions generate $(10M+10N+16)$, $(10M+10N+18)$, $(10M+10N+15)$ and $(10M+10N+10)$ algebraic equations respectively. It is clear that the total number of equations is more than the unknown coefficients. In order to have a compatible solution, the multiple regression analysis [14] based on the least-squares error norms is used. The non-linear terms are transferred to the right side and computed at each step of marching variable. The left-side matrix consists of linear terms and hence remains invariant with respect to the marching variable. The set of linear equations are expressed in matrix form as

$$[A]\{a\} = \{Q\}, \tag{26}$$

where $[A]$ is the $(M \times N)$ coefficient matrix, $\{a\}$ is the $(N \times 1)$ displacement vector and $\{Q\}$ is the $(M \times 1)$ load vector. Multiple regression analysis [13] gives

$$\{a\} = ([A]^{\mathrm{T}}[A])^{-1}[A]^{\mathrm{T}}\{Q\} \tag{27}$$

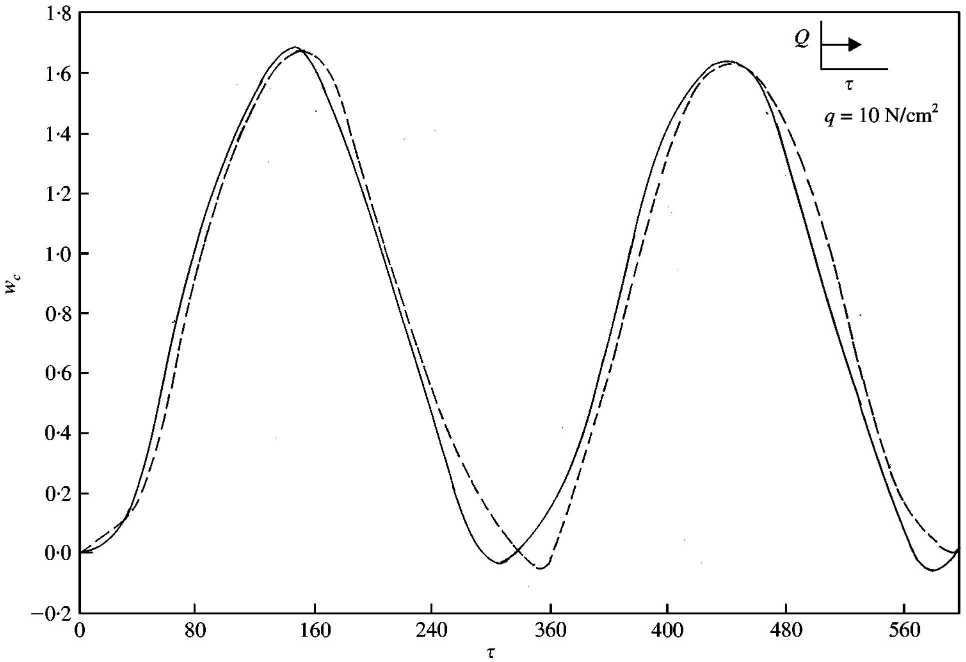

Figure 2. Comparison of central displacement response for simply supported thick isotropic square plate ($\nu = 0{\cdot}3$, $a/h = 5$) under step loading: ——, present study; — — —, Reddy and Chandrashekhara [19].

or

$$\{a\} = [B]\{Q\}. \tag{28}$$

The matrix $[B]$ is evaluated once and retained for subsequent usage.

4. RESULTS AND DISCUSSION

In the present study the non-linear governing equations of motion of laminated composite rectangular plates subjected to step, ramp and sinusoidal loading, respectively, are solved using the fast-converging Chebyshev polynomials and Houbolt time-marching scheme. In order to check the accuracy and stability of the method, convergence study was conducted. It can be noted from Table 1 that 9–11 terms expansion of Chebyshev series and an increment of 0·1 for time τ are sufficient to yield quite accurate results. An iterative methodology of solution with relative convergence criteria of 0·01% of each coefficient at every step is employed. The present methodology is validated with the results of Reddy and Chandrshekhara [19], obtained using the FEM for a simply supported isotropic square plate

$$(a = 25\,\mathrm{cm},\ h = 5\,\mathrm{cm},\ E = 2{\cdot}1\times10^{6}\,\mathrm{N/cm^2},\ \nu = 0{\cdot}25,\ \rho = 8\times10^{-6}\,\mathrm{N\,s^2/cm^4})$$

under uniformly distributed load and the comparison is shown in Figure 2.

The displacement response results of the laminated composite plate with non-Levy-type boundary conditions, i.e., CCCF, CCFF and CSCS are presented for material properties $E_1/E_2 = 25$, $G_{12} = 0{\cdot}5E_2$, $G_{23} = 0{\cdot}2E_2$, $G_{13} = G_{12}$, $\nu_{12} = 0{\cdot}25$. The influence of various

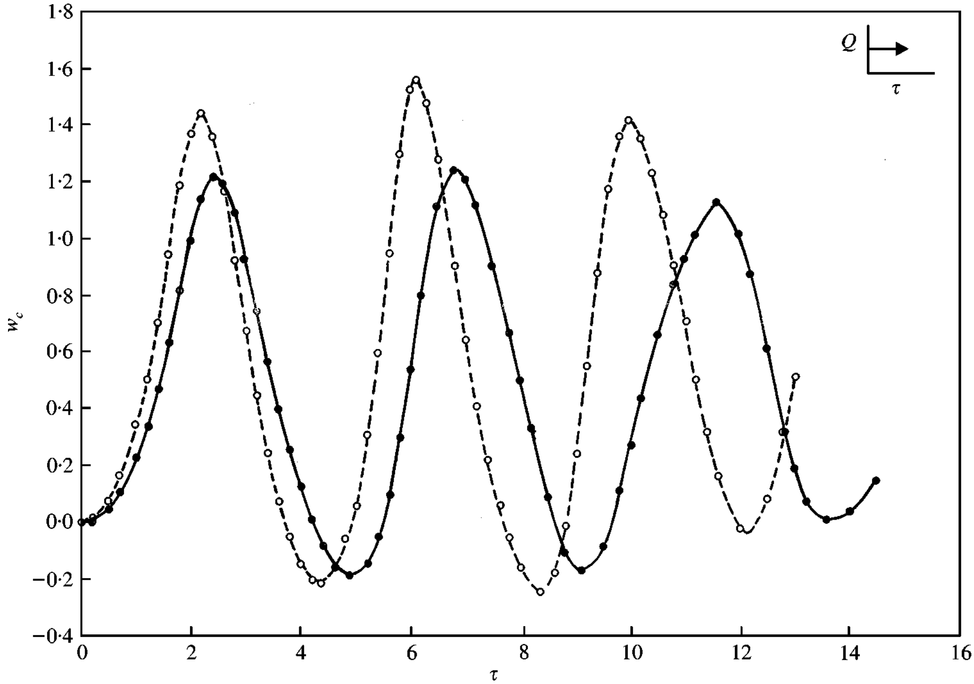

Figure 3. Displacement response of anitsymmetric angle-ply [15/ − 15/15/ − 15] square CSCS plate ($a/h = 10$) under step loading: --○--, $Q = 375$; —●—, $Q = 250$.

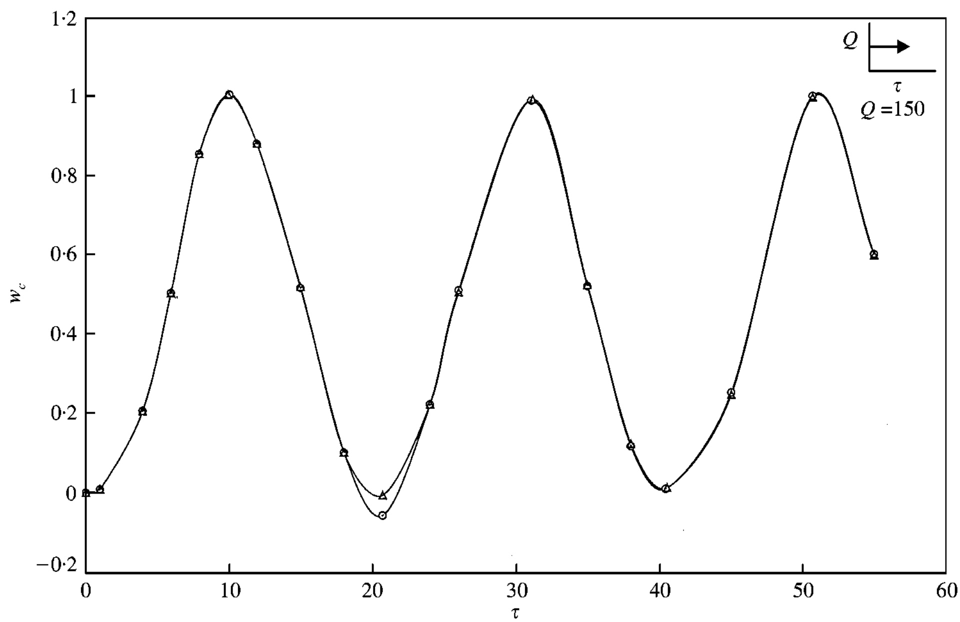

Figure 4. Effect of inertia on central displacement response for symmetrically laminated cross-ply [0/90/90/0] square CCFF plate ($a/h = 10$) under step loading: ——, all inertia; —○—, in-plane + lateral, —△—, lateral.

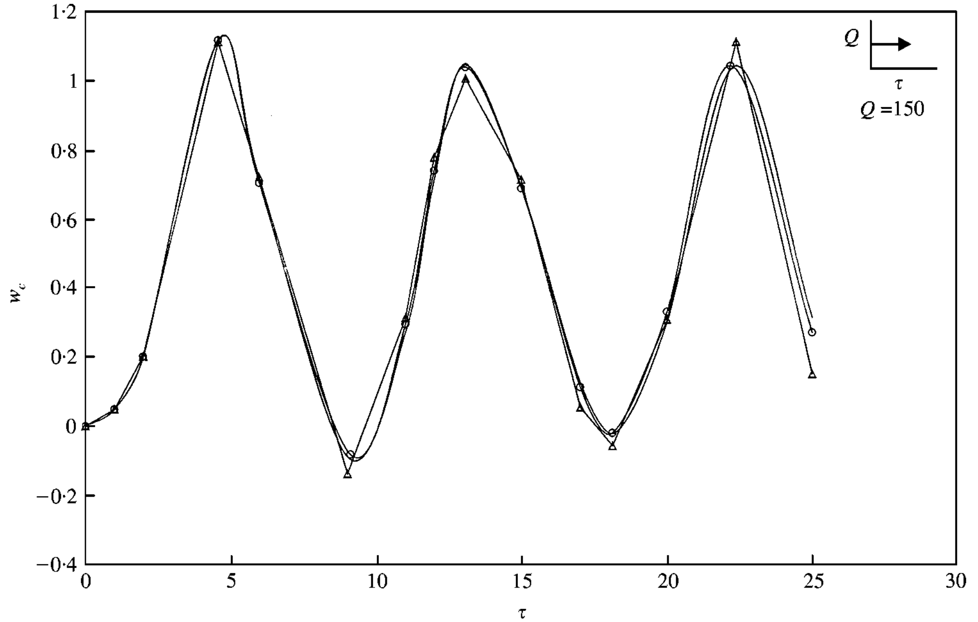

Figure 5. Effect of inertia on central displacement response for unsymmetrically laminated angle-ply [0/15/30/45] square CSCS plate ($a/h = 10$) under step loading: ——, all inertia; —○—, in-plane + lateral; —△—, lateral.

inertia, lamination scheme, aspect ratio, span-to-thickness ratio, boundary conditions, magnitude and duration of the pulse loading on the transient response is analyzed.

The response of the antisymmetric angle-ply [15/ − 15/15/ − 15] square plate ($a/h = 10$) with CSCS boundary condition and subjected to step loadings $Q = 250$ and 375 is plotted in Figure 3. It is clear from the results that the magnitude and frequency of the non-linear response increases with increase in step loading. This hardening-type behaviour is because bending stiffness of the plate increases with increase in lateral deflection. The effects of in-plane, lateral, coupled normal-rotary and rotary inertia on the central displacement response have been investigated for cross-ply symmetrically laminated [0/90/90/0] square CCFF plate ($a/h = 10$, $Q = 150$) and [0/15/30/45] unsymmetrically laminated angle-ply square CSCS plates ($a/h = 10$, $Q = 150$) and are shown in Figures 4 and 5 respectively. It can be observed that the influence of in-plane inertia and rotary inertia is insignificant.

The influence of b/a on the central displacement response of antisymmetric cross-ply [0/90/0/90] laminated CCCF plate ($a/h = 10$), under step load $Q = 200$ is plotted in Figure 6. Central response is minimum for $b/a = 0{\cdot}5$ and maximum for $b/a = 5{\cdot}0$. It is interesting to note that deflection response for $b/a = 2$ and 3 is lower than that of $b/a = 1{\cdot}5$. The influence of a/h on the central displacement response has been investigated and results for antisymmetric cross-ply [0/90/0/90] CSCS square plate are shown in Figure 7 for step loading $Q = 250$. It is interesting to note that the maximum central deflection is almost the same for $a/h = 30$, 40 and 50. It is due to the fact that for $a/h > 20$, the plates belong to thin plate category. The response is maximum for $a/h = 10$ and there is a significant difference as compared to $a/h = 30$. The time period increases with increase in a/h.

The influence of boundary conditions on the central response has been studied and the results for four-layer cross-ply antisymmetric SSSS, CSCS, CCFF and CCCF plates are plotted in Figure 8 for step load $Q = 125$, $a/h = 10$ and $b/a = 1$. It can be observed that

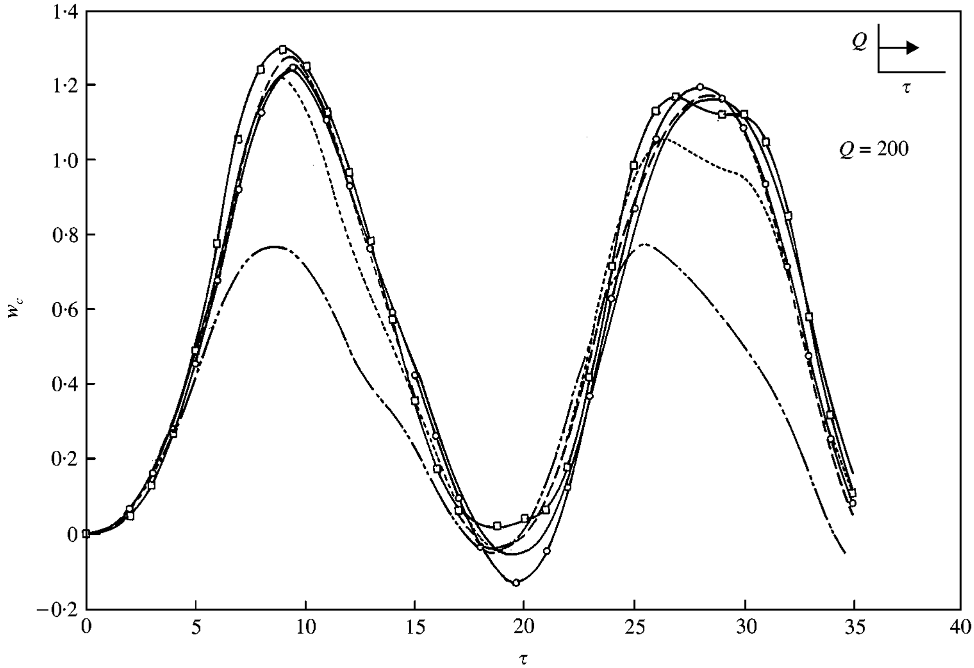

Figure 6. Effect of b/a on central displacement response for antisymmetric laminated cross-ply [0/90/0/90] square CCCF plate ($a/h = 10$) under step loading: b/a; ——·——, 0·5; -------, 1·0; – – – –, 1·5; —○—, 2·0; ——, 3·0; —□—, 5·0.

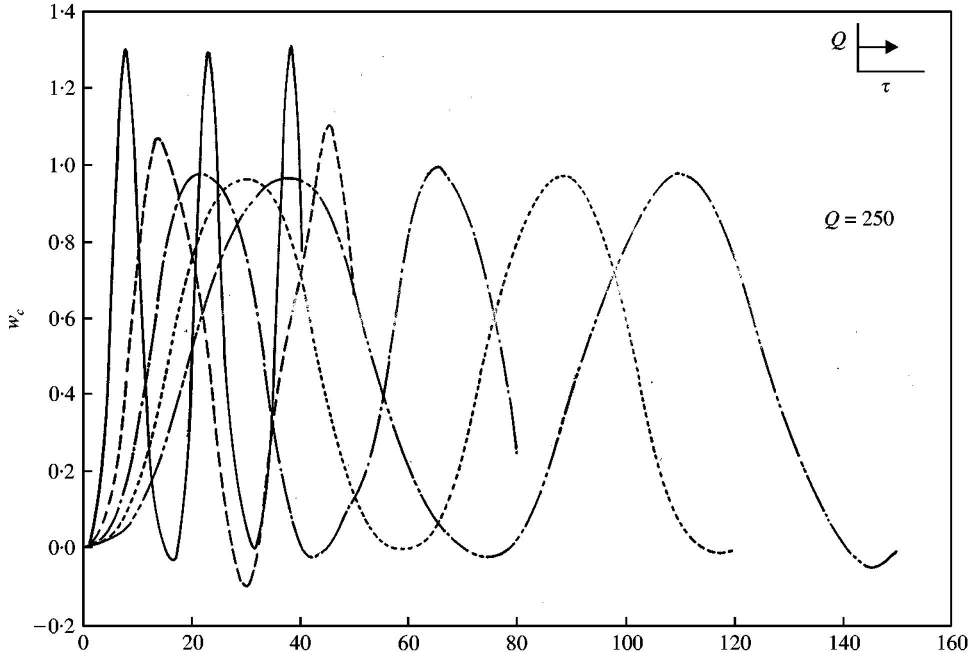

Figure 7. Effect of a/h on the central displacement response for antisymmetric laminated cross-ply [0/90/0/90] square CSCS plate under step loading: a/h; ———, 10; – – –, 20; —·—, 30; -------, 40; — – —, 50.

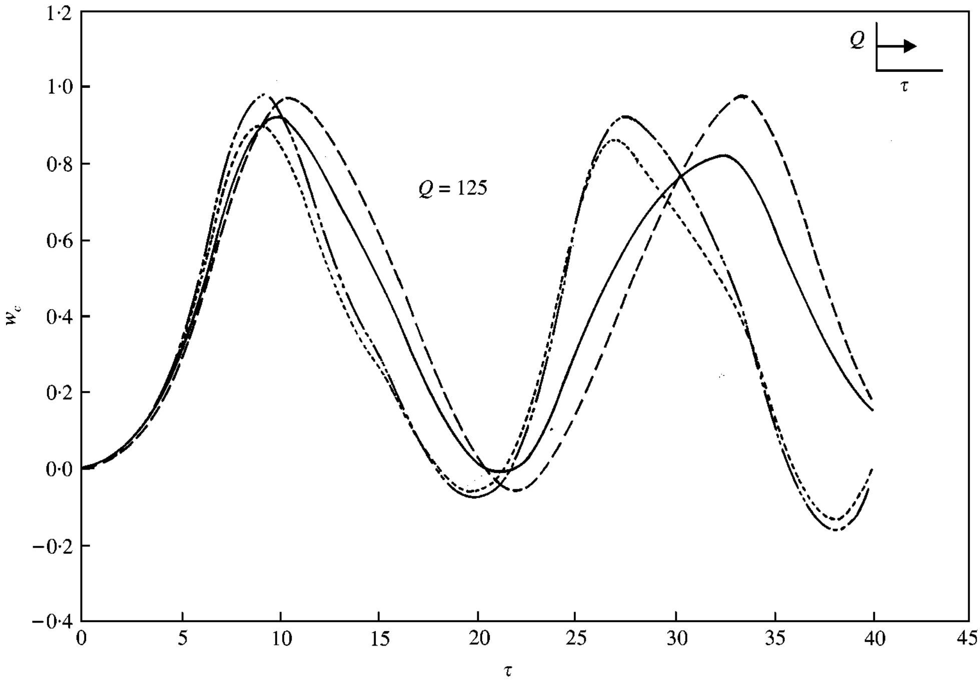

Figure 8. Effect of boundary conditions on central displacement response for antisymmetric laminated cross-ply [0/90/0/90] square plate ($a/h = 10$) under step loading: ——, CCCF, ----, CCFF; ------, CSCS; — -- —, SSSS.

the maximum response occurs for simply supported SSSS plate and the minimum for CSCS plate. Also, it is interesting to note that the central response of CCFF plate is lesser than SSSS plate in the first half-cycle and the reverse trend is observed in the next half-cycle. The effects of lamination scheme on the transient response of CCCF square plate for $a/h = 10$ and step load $Q = 100$ are shown in Figure 9. The responses are lower for symmetric $[22{\cdot}5/-22{\cdot}5/-22{\cdot}5/22{\cdot}5]$ and antisymmetric $[22{\cdot}5/-22{\cdot}5/22{\cdot}5/-22{\cdot}5]$ angle-ply plates as compared to other lamination schemes. The response is higher for the unsymmetric laminated plate [0/15/30/45]. The symmetric [0/90/90/0] and the antisymmetric [0/90/0/90] cross-ply plates have the lowest frequencies. This is attributed to material coupling coefficients.

The influence of duration of step function and sinusoidal loading on the central response has been studied. The results of the four-layer antisymmetric cross-ply CSCS plate subjected to sinusoidal loading are shown in Figure 10 for pulse durations of 5 and 10 respectively. It is noted that transient-free response increases with the duration of the pulse. The response for four-layer antisymmetric cross-ply CSCS square plate ($a/h = 10$) subjected to step loading to infinite duration and ramp of pulse durations of 5 and 10 are shown in Figure 11. The response of step loading for infinite duration is higher and for ramp pulse of duration 10 it is almost similar. The response of unsymmetric angle-ply [0/15/30/45] CSCS square plate ($a/h = 10$) is shown in Figure 12. It is interesting to observe that response to step loading of infinite duration and $T_P = 5$ have almost the same amplitude. Surprisingly, the amplitude of response for $T_P = 10$ is less than that of $T_P = 5$. The response to sinusoidal loading of duration $T_P = 5$ and 10 is also shown in this figure for comparison purpose. The amplitude of the response to a sinusoidal pulse of $T_P = 5$ duration is lowest and it is highest

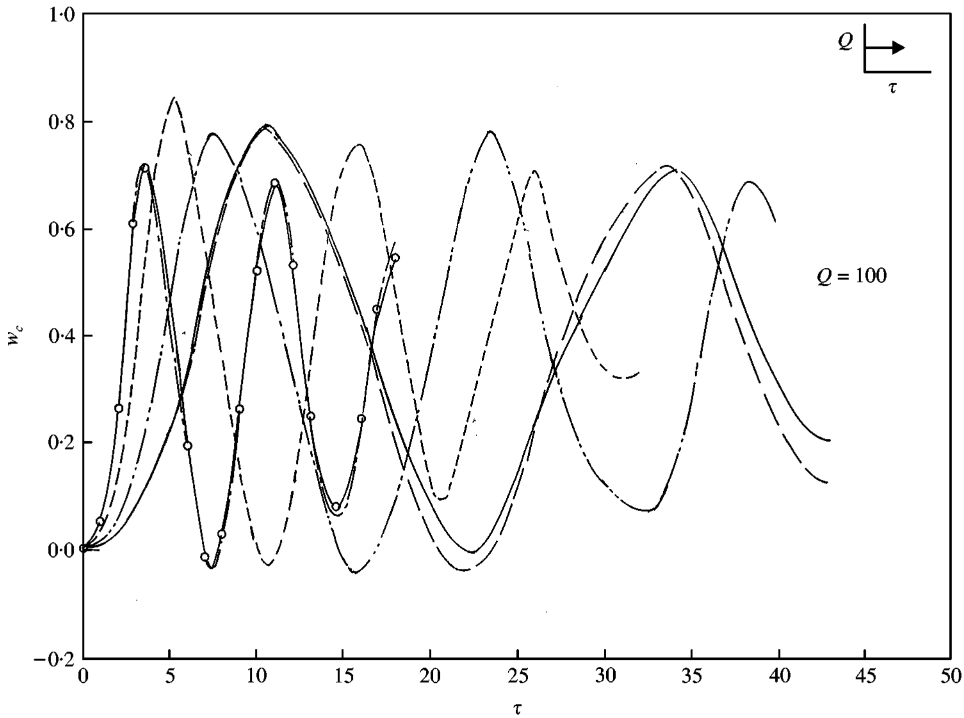

Figure 9. Effect of lamination schemes on central displacement response for square CCCF plate ($a/h = 10$) under step loading: — —, [0/90/90/0]; ——, [0/90/0/90]; ------, [0/15/30/45]; —○—, [± 22·5/ ± 22·5]; — - —, [± 22·5]$_s$; — ·· —, [0/0/0/90].

for $T_P = 10$. This observation is not in line with the common notion that step function loading generates higher response as compared to sinusoidal pulses having the same magnitude and duration.

5. CONCLUSIONS

Based on Chebyshev approximation, an analytical solution to the non-linear transient analysis of moderately thick laminated composite plates with SSSS, CSCS, CCCF and CCFF boundary conditions is presented. The effects of in-plane inertia, rotary inertia and coupled normal-rotary inertia on the response are insignificant. Central response is minimum for $b/a = 0{\cdot}5$ and maximum for $b/a = 5{\cdot}0$. The effect of b/a on the response remains significant up to $b/a = 1{\cdot}5$, beyond which there is no appreciable difference in the response. The effect of a/h on the displacement response is noticeable up to $a/h = 20$ and beyond $a/h = 30$ it has no significant effect. The central displacement response of unsymmetric laminate is higher for step, ramp and sinusoidal loadings as compared to symmetric or antisymmetric laminate. Transient-free response increases with increase in the duration of the pulse loading. Response to a step loading is lower than that to sinusoidal loading. It can be observed that the present method can be efficiently applied for the non-linear transient analysis of laminated composite plates with complicated boundary conditions.

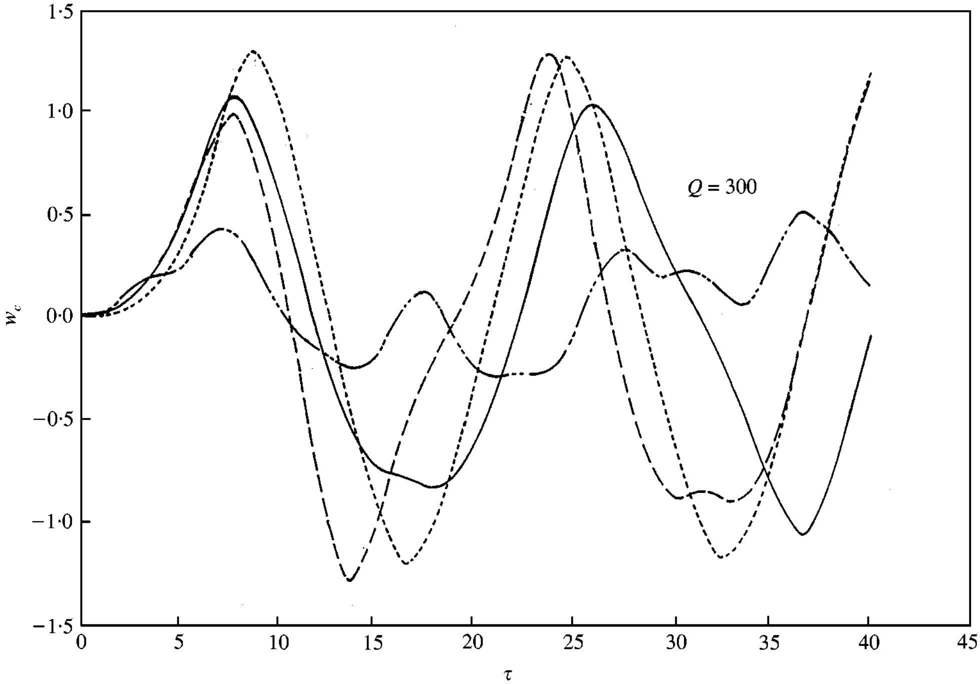

Figure 10. Effect of duration of sinusoidal loading on central displacement response for antisymmetric laminated cross-ply [0/90/0/90] square CSCS plate ($a/h = 10$). : — -- —, $T_p = 5$; - - - -, $T_p = 10$. : ———, $T_p = 5$; -------, $T_p = 10$.

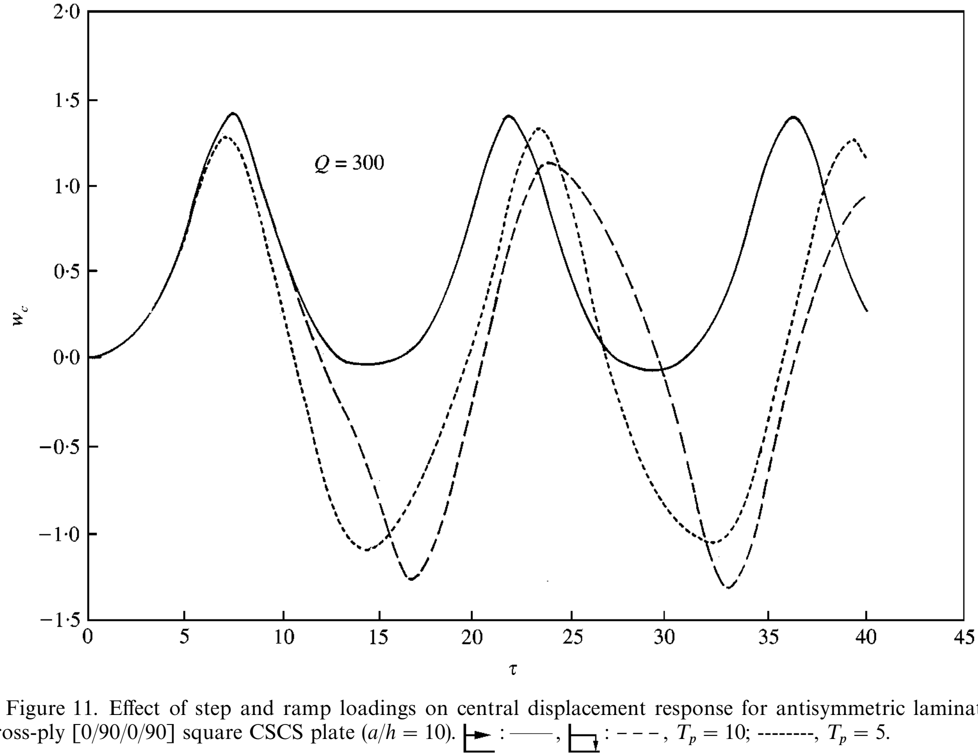

Figure 11. Effect of step and ramp loadings on central displacement response for antisymmetric laminated cross-ply [0/90/0/90] square CSCS plate ($a/h = 10$). : ——, : – – –, $T_p = 10$; --------, $T_p = 5$.

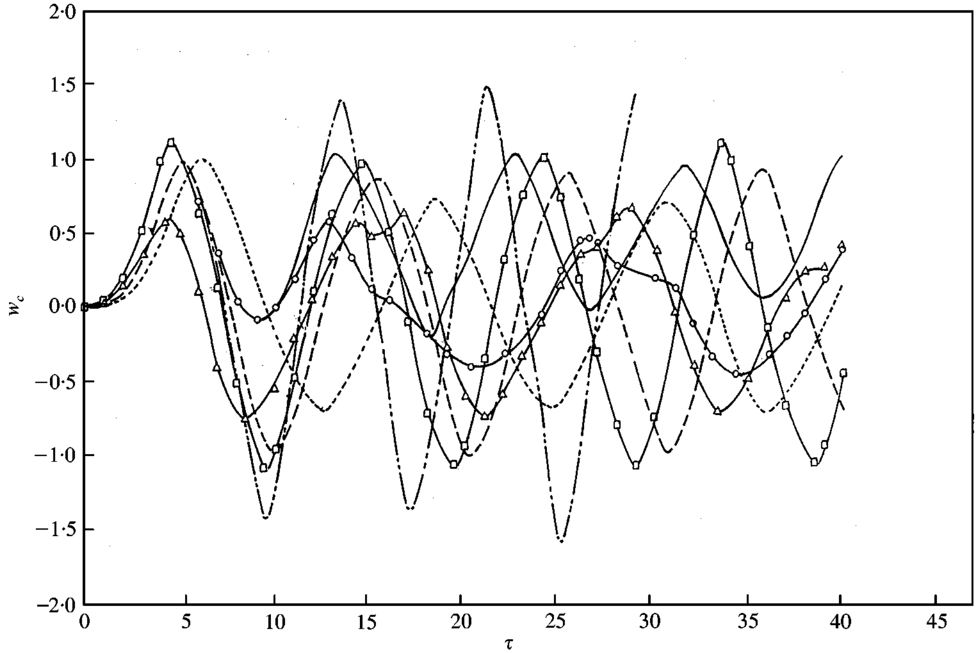

Figure 12. Effect of different loadings (step, ramp and sinusoidal) on central displacement response for antisymmetric laminated cross-ply [0/15/30/45] square CSCS plate ($a/h = 10$). : ——. : —□—, 5; —○—, 10. : – – –, 5; -----, 10. : —△—, 5; — - - —, 10.

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APPENDIX A: DEFINITION OF NON-DIMENSIONAL PARAMETERS AND COEFFICIENTS

$$\lambda = \frac{a}{b}, \quad \beta = \frac{a}{h}, \qquad x = \frac{2X}{a}, y = \frac{2Y}{b}, u = \frac{u_0}{h}, v = \frac{v_0}{h}, w = \frac{w_0}{h}, \quad \bar{N}_x = \frac{N_x\beta}{A_{11}}, \bar{N}_y = \frac{N_y\beta}{A_{22}},$$

$$\bar{N}_{xy} = \frac{N_{xy}\beta}{A_{66}}, \bar{M}_x = \frac{M_x h\beta^2}{D_{11}}, \quad \bar{M}_y = \frac{M_y h\beta^2}{D_{22}}, \bar{M}_{xy} = \frac{M_{xy} h\beta^2}{D_{66}}, q^* = \frac{qa^2}{4hA_{22}}, Q = \frac{qa^4}{E_2 h^4},$$

$$\tau = t \sqrt{\frac{4A_{22}}{Ph^2\beta^2}}.$$

$$L_1 = \frac{A_{66}\lambda^2}{A_{11}}, L_2 = \frac{2A_{16}\lambda}{A_{11}}, L_3 = \frac{A_{16}}{A_{11}}, L_4 = \frac{A_{26}\lambda^2}{A_{11}}, L_5 = \frac{(A_{12}+A_{66})\lambda}{A_{11}}, L_6 = \frac{B_{11}}{A_{11}h},$$

$$L_7 = \frac{B_{66}\lambda^2}{A_{11}h}, L_8 = \frac{2B_{16}\lambda}{A_{11}h}, L_9 = \frac{B_{16}}{A_{11}h}, L_{10} = \frac{B_{26}\lambda^2}{A_{11}h}, L_{11} = \frac{(B_{12}+B_{66})\lambda}{A_{11}h}, L_{12} = \frac{2}{\beta},$$

$$L_{13} = \frac{2A_{66}\lambda^3}{A_{11}\beta}, L_{14} = \frac{4A_{16}\lambda}{A_{11}\beta}, L_{15} = \frac{2A_{16}\lambda}{A_{11}\beta}, L_{16} = \frac{2A_{26}\lambda^3}{A_{11}\beta}, L_{17} = \frac{2(A_{12}+A_{66})\lambda^2}{A_{11}\beta},$$

$$L_{18} = \frac{A_{16}}{A_{22}}, L_{19} = \frac{A_{26}\lambda^2}{A_{22}}, L_{20} = \frac{(A_{12}+A_{66})\lambda}{A_{22}}, L_{21} = \frac{A_{66}}{A_{22}}, L_{22} = \lambda^2, L_{23} = \frac{2A_{26}\lambda}{A_{22}},$$

$$L_{24} = \frac{B_{16}}{A_{22}h}, L_{25} = \frac{B_{26}\lambda^2}{A_{22}h}, L_{26} = \frac{(B_{12}+B_{66})\lambda}{A_{22}h}, L_{27} = \frac{B_{66}}{A_{22}h}, L_{28} = \frac{B_{22}\lambda^2}{A_{22}h}, L_{29} = \frac{2B_{26}\lambda}{A_{22}h},$$

$$L_{30} = \frac{2A_{16}}{A_{22}\beta}, L_{31} = \frac{2A_{26}\lambda^2}{A_{22}\beta}, L_{32} = \frac{2(A_{12}+A_{66})\lambda}{A_{22}\beta}, L_{33} = \frac{2A_{66}\lambda}{A_{22}\beta}, L_{34} = \frac{2\lambda^3}{\beta}, L_{35} = \frac{2A_{26}\lambda^2}{A_{22}\beta},$$

$$L_{36} = \frac{A_{55}}{A_{22}}, L_{37} = \frac{A_{44}\lambda^2}{A_{22}}, L_{38} = \frac{2A_{45}\lambda}{A_{22}}, L_{39} = \frac{A_{55}\beta}{2A_{22}}, L_{40} = \frac{A_{45}\lambda\beta}{2A_{22}}, L_{41} = \frac{A_{45}\beta}{2A_{22}},$$

$$L_{42} = \frac{A_{44}\lambda\beta}{2A_{22}}, L_{43} = \frac{2A_{11}}{A_{22}\beta}, L_{44} = \frac{2A_{16}\lambda}{A_{22}\beta}, L_{45} = \frac{2A_{16}}{A_{22}\beta}, L_{46} = \frac{2A_{12}\lambda}{A_{22}\beta}, L_{47} = \frac{2B_{11}}{A_{22}h\beta},$$

$$L_{48}=\frac{2B_{16}\lambda}{A_{22}h\beta}, L_{49}=\frac{2B_{16}}{A_{22}h\beta}, L_{50}=\frac{2B_{12}\lambda}{A_{22}h\beta}, L_{51}=\frac{2A_{12}\lambda^2}{A_{22}\beta}, L_{52}=\frac{2A_{26}\lambda^3}{A_{22}\beta}, L_{53}=\frac{2A_{26}\lambda^2}{A_{22}\beta},$$

$$L_{54}=\frac{2\lambda^3}{\beta}, L_{55}=\frac{2B_{12}\lambda^2}{A_{22}h\beta}, L_{56}=\frac{2B_{26}\lambda^3}{A_{22}h\beta}, L_{57}=\frac{2B_{26}\lambda^2}{A_{22}h\beta}, L_{58}=\frac{2B_{22}\lambda^3}{A_{22}h\beta}, L_{59}=\frac{2A_{16}\lambda}{A_{22}\beta},$$

$$L_{60}=\frac{2A_{66}\lambda^2}{A_{22}\beta}, L_{61}=\frac{4A_{16}\lambda}{A_{22}\beta}, L_{62}=\frac{4A_{26}\lambda^2}{A_{22}\beta}, L_{63}=\frac{4B_{16}\lambda}{A_{22}h\beta}, L_{64}=\frac{4B_{66}\lambda^2}{A_{22}h\beta}, L_{65}=\frac{4B_{66}\lambda}{A_{22}h\beta},$$

$$L_{66}=\frac{4B_{26}\lambda^2}{A_{22}h\beta}, L_{67}=\frac{2A_{11}}{A_{22}\beta^2}, L_{68}=\frac{2A_{12}\lambda^2}{A_{22}\beta^2}, L_{69}=\frac{2A_{16}\lambda}{A_{22}\beta^2}, L_{70}=\frac{2A_{12}\lambda^2}{A_{22}\beta^2}, L_{71}=\frac{2\lambda^4}{\beta^2},$$

$$L_{72}=\frac{4A_{26}\lambda^3}{A_{22}\beta^2}, L_{73}=\frac{4A_{16}\lambda}{A_{22}\beta^2}, L_{74}=\frac{4A_{26}\lambda^3}{A_{22}\beta^2}, L_{75}=\frac{8A_{66}\lambda}{A_{22}\beta^2},$$

$$L_{76}=\frac{B_{11}h}{D_{11}}, L_{77}=\frac{B_{66}h\lambda^2}{D_{11}}, L_{78}=\frac{2B_{16}h\lambda}{D_{11}}, L_{79}=\frac{B_{16}h}{D_{11}}, L_{80}=\frac{B_{26}h\lambda^2}{D_{11}}, L_{81}=\frac{(B_{12}+B_{66})h\lambda}{D_{11}},$$

$$L_{82}=\frac{D_{66}\lambda^2}{D_{11}}, L_{83}=\frac{2D_{16}\lambda}{D_{11}}, L_{84}=\frac{D_{16}}{D_{11}}, L_{85}=\frac{D_{26}\lambda^2}{D_{11}}, L_{86}=\frac{(D_{12}+D_{66})\lambda}{D_{11}}, L_{87}=-\frac{A_{55}h^2\beta}{2D_{11}},$$

$$L_{88}=-\frac{A_{45}h^2\lambda\beta}{2D_{11}}, L_{89}=-\frac{A_{55}h^2\beta^2}{4D_{11}}, L_{90}=-\frac{A_{45}h^2\beta^2}{4D_{11}}, L_{91}=\frac{2B_{11}h}{D_{11}\beta}, L_{92}=\frac{2B_{66}h\lambda^2}{D_{11}\beta},$$

$$L_{93}=\frac{4B_{16}h\lambda}{D_{11}\beta}, L_{94}=\frac{2B_{16}h\lambda}{D_{11}\beta}, L_{95}=\frac{2B_{26}h\lambda^3}{D_{11}\beta}, L_{96}=\frac{2(B_{12}+B_{66})h\lambda^2}{D_{11}\beta},$$

$$L_{97}=\frac{B_{16}h}{D_{22}}, L_{98}=\frac{B_{26}h\lambda^2}{D_{22}}, L_{99}=\frac{(B_{12}+B_{66})h\lambda}{D_{22}}, L_{100}=\frac{B_{66}h}{D_{22}}, L_{101}=\frac{B_{22}h\lambda^2}{D_{22}},$$

$$L_{102}=\frac{2B_{26}h\lambda}{D_{22}}, L_{103}=\frac{D_{16}}{D_{22}}, L_{104}=\frac{D_{26}\lambda^2}{D_{22}}, L_{105}=\frac{(D_{12}+D_{66})\lambda}{D_{22}}, L_{106}=\frac{D_{66}}{D_{22}}, L_{107}=\lambda^2,$$

$$L_{108}=\frac{2D_{26}\lambda}{D_{22}}, L_{109}=-\frac{A_{45}h^2\beta}{2D_{22}}, L_{110}=-\frac{A_{44}h^2\lambda\beta}{2D_{22}}, L_{111}=-\frac{A_{45}h^2\beta^2}{4D_{22}}, L_{112}=-\frac{A_{44}h^2\beta^2}{4D_{22}},$$

$$L_{113}=\frac{2B_{16}h}{D_{22}\beta}, L_{114}=\frac{2B_{26}h\lambda^2}{D_{22}\beta}, L_{115}=\frac{2(B_{12}+B_{66})h\lambda}{D_{22}\beta}, L_{116}=\frac{2B_{66}h\lambda}{D_{22}\beta}, L_{117}=\frac{2B_{22}h\lambda^3}{D_{22}\beta},$$

$$L_{118}=\frac{4B_{26}h\lambda^2}{D_{22}\beta}.$$

$$F_1=\frac{A_{22}}{A_{11}}, F_2=\frac{A_{22}R}{A_{11}Ph}, F_3=\frac{R}{Ph}, F_4=\frac{A_{22}Rh}{D_{11}P}, F_5=\frac{A_{22}I}{D_{11}P}, F_6=\frac{A_{22}Rh}{D_{22}P}, F_7=\frac{A_{22}I}{D_{22}P}.$$

APPENDIX B

HOUBOLT ACCELERATION COEFFICIENTS FOR STEP FUNCTION LOAD

Time step	β_1	β_2	β_3	β_4	β_5
$J=1$	6	0	0	0	$-2Q\Delta\tau^2$
$J=2$	2	-4	0	0	$-Q\Delta\tau^2$
$J=3$	2	-5	4	0	0
$J>3$	2	-5	4	-1	0

HOUBOLT ACCELERATION COEFFICIENTS FOR SINUSOIDAL PULSE LOAD

Time step	β_1	β_2	β_3	β_4	β_5
$J=1$	6	0	0	0	0
$J=2$	2	-4	0	0	0
$J=3$	2	-5	4	0	0
$J>3$	2	-5	4	-1	0

APPENDIX C: NOMENCLATURE

a, b, h	plate dimensions
U, V, W	displacements along X, Y and Z directions
u_0, v_0, w_0	displacements of mid-plane
u, v, w	non-dimensional displacements of mid-plane
ε_x, ε_y, γ_{xy}, γ_{xz}, γ_{yz}	strain components
w_c, Q, τ	non-dimensional central deflection, load, time
ψ_x, ψ_y	slopes in xz and yz plane respectively
M, N	number of terms in Chebyshev series in x and y directions respectively
N_x, N_y, N_{xy}	stress resultants
M_x, M_y, M_{xy}	moment resultants
$\bar{N}_x$, $\bar{N}_y$, $\bar{N}_{xy}$	non-dimensional stress resultants
$\bar{M}_x$, $\bar{M}_y$, $\bar{M}_{xy}$	non-dimensional moment resultants
Q_x, Q_y	shear forces
$\bar{Q}_x$, $\bar{Q}_y$	non-dimensional shear forces
SSSS	all edges simply supported
CSCS	two adjacent edges ($x=-1$ and $y=1$) clamped and two edges ($x=+1$ and $y=-1$) simply supported
CCCF	three edges ($x=\pm 1$ and $y=1$) clamped and one edge ($y=-1$) free
CCFF	two opposite edges ($x=\pm 1$) clamped and two edges ($y=\pm 1$) free
$T(x)$, $T(y)$	Chsebyshev polynomials
( ),	comma denotes the differentiation with respect to the variables that follow